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Viscous flows past spherical gas bubbles

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Computations of the steady viscous flow past a fixed spherical gas bubble are reported for Reynolds numbers in the range $0{\cdot}1 \leqslant R \leqslant 200$. Good agreement with Moore's (1963) asymptotic theory for the drag coefficient is obtained for $R \geqslant 40$ and with the well-known small-R theory for $R \leqslant \frac{1}{2}$. The method of series truncation is used to reduce the problem to a nonlinear two-point boundary-value problem, which is then solved by an accurate and efficient finite-difference procedure.

1. Introduction

We have computed the steady, three-dimensional, axisymmetric, incompressible, viscous flow past a fixed spherical gas bubble. Our main goal is to check Moore's (1963) asymptotic theory for high Reynolds numbers and to fill in the gap down to small Reynolds numbers where $C_D = 8/R$ is known to be valid. To do this we have developed an efficient accurate computation scheme for moderate Reynolds numbers using the method of series truncation. Our method is easily adapted to flows past a rigid sphere and we have obtained close agreement with Dennis & Walker (1971) for this case.

Several methods have been used for calculating flows past rigid spheres. Jenson (1959) used finite differences and relaxation methods on the stream-function and vorticity equations. Rimon & Cheng (1969) obtained steady solutions by integrating the time-dependent equations to a steady state. Dennis & Walker (1971) used the method of series truncation. This method has also been used for flows about rigid cylinders by Underwood (1969) and by Keller & Nieuwstadt (1973).

Here we formulate the Navier–Stokes equations in terms of a stream function and the vorticity and then seek solutions in the form of series of Legendre functions. This procedure yields a system of coupled second-order nonlinear differential equations for the coefficients. Up to this point our work differs from that of Dennis & Walker (1971) only in the boundary conditions. We then solve this system using the centred Euler or 'box' scheme analysed by Keller (1974). We use Newton's method to solve the resulting difference equations and Richardson extrapolation to improve the accuracy. This scheme has also been used by Keller & Nieuwstadt (1973) for flows about cylinders.

Our calculations were carried out for the Reynolds numbers $R = 0{\cdot}1$, $0{\cdot}5$, 1, 5, 10, 20, 40, 60, 120 and 200. The results agree well with Moore's theory when $R \geqslant 40$ and with the small-R theory when $R \leqslant \frac{1}{2}$.

2. Formulation

Since the flow is axisymmetric we use (r, θ) co-ordinates, where r is the distance from the sphere centre and θ is the angle measured counterclockwise from the downstream direction. We define a dimensionless radial variable ξ by

$$e^{\xi} = r/a, \tag{2.1}$$

where a is the sphere radius. The radial and transverse velocity components (u, v) are defined in terms of a stream function ψ by

$$u = \frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v = -\frac{e^{-2\xi}}{\sin\theta}\frac{\partial\psi}{\partial\xi} \tag{2.2a}$$

and then the vorticity ζ is determined from

$$e^{\xi}\zeta = \partial v/\partial\xi + v - \partial u/\partial\theta. \tag{2.2b}$$

The Navier–Stokes equations become

$$\frac{\partial^2\zeta}{\partial\xi^2} + \frac{\partial\zeta}{\partial\xi} + \cot\theta\frac{\partial\zeta}{\partial\theta} + \frac{\partial^2\zeta}{\partial\theta^2} - \frac{\zeta}{\sin^2\theta} = \tfrac{1}{2}Re^{\xi}\left(u\frac{\partial\zeta}{\partial\xi} + v\frac{\partial\zeta}{\partial\theta} - u\zeta - v\zeta\cot\theta\right), \tag{2.3}$$

where $R = 2aU/\nu$ is the Reynolds number, U is the free-stream speed and ν is the kinematic viscosity. As $\xi \to \infty$ we impose the Oseen expansion for flow about a rigid sphere as a boundary condition. This is actually done at a finite radius ξ_∞. Some discussion of this procedure is given in §4. On the gas-bubble surface, we impose the zero stress condition. (For a rigid sphere this would be replaced by the no-slip condition.) Thus, the boundary conditions are (recalling the symmetry about $\theta = 0$ and $\theta = \pi$)

$$\zeta = \psi = 0 \quad \text{on} \quad \theta = 0, \pi, \tag{2.4a}$$

$$\psi = 0, \quad e^{-\xi}\frac{\partial u}{\partial\theta} + \frac{\partial}{\partial\xi}(e^{-\xi}v) = 0 \quad \text{on} \quad \xi = 0, \tag{2.4b}$$

$$\left.\begin{aligned} \psi(\xi_\infty, \theta) &= \tfrac{1}{2}e^{2\xi}\sin^2\theta + \tfrac{1}{4}e^{-\xi}\sin^2\theta - (3/R)(1+\cos\theta) \\ &\qquad \times\{1 - \exp[-\tfrac{1}{4}Re^{\xi}(1-\cos\theta)]\}, \\ \zeta(\xi_\infty, \theta) &= -\tfrac{3}{2}e^{-2\xi}\sin\theta\exp[-\tfrac{1}{4}Re^{\xi}(1-\cos\theta)](1+\tfrac{1}{4}Re^{\xi}). \end{aligned}\right\} \tag{2.4c}$$

We now seek series expansions of ψ and ζ of the form

$$\psi(\xi, \theta) = e^{\frac{1}{2}\xi}\sum_{n=1}^{\infty} f_n(\xi)\int_z^1 P_n(t)\,dt, \tag{2.5a}$$

$$\zeta(\xi, \theta) = \sum_{n=1}^{\infty} g_n(\xi)\,P_n^{(1)}(z), \tag{2.5b}$$

where $z = \cos\theta$ and $P_n(z)$ and $P_n^{(1)}(z)$ are, respectively, the Legendre polynomial and first associated Legendre function of order n. Substituting these expansions into (2.2) and (2.3) we obtain for $n = 1, 2, \ldots$,

$$f_n''(\xi) - (n+\tfrac{1}{2})^2 f_n(\xi) = e^{\frac{3}{2}\xi} n(n+1)\, g_n(\xi), \tag{2.6a}$$

$$g_n''(\xi) + g_n'(\xi) - n(n+1)\, g_n(\xi) = r_n, \tag{2.6b}$$

$$r_n = \tfrac{1}{2} Re^{-\frac{1}{2}\xi} \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} \{\alpha_{m,l}^n f_m(\xi)\,[g_l'(\xi) - g_l(\xi)] + \beta_{m,l}^n g_l(\xi)\,[f_m'(\xi) + \tfrac{1}{2} f_m(\xi)]\},$$

where $$\alpha_{m,l}^n = -(2n+1)\left[\frac{l(l+1)}{n(n+1)}\right]^{\frac{1}{2}} \begin{pmatrix} n & m & l \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} n & m & l \\ 0 & 0 & 0 \end{pmatrix},$$

$$\beta_{m,l}^n = -(2n+1)\left[\frac{l(l^2-1)(l+2)}{nm(n+1)(m+1)}\right]^{\frac{1}{2}} \begin{pmatrix} n & m & l \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} n & m & l \\ 0 & 0 & 0 \end{pmatrix}.$$

Here the $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$ are the 3-j symbols described by Rotenberg *et al.* (1959). The boundary conditions on the f_n and g_n are

$$\left.\begin{aligned} f_n(0) = 0, \quad f_n'(0) - \tfrac{1}{2} n(n+1)\, g_n(0) = 0, \\ f_n(\xi_\infty) = f_n^\infty(\xi_\infty), \quad g_n(\xi_\infty) = g_n^\infty(\xi_\infty), \end{aligned}\right\} \quad n = 1, 2, \ldots. \tag{2.7}$$

Here $f_n^\infty(\xi)$ and $g_n^\infty(\xi)$ are determined by using (2.5) in the Oseen expansion (2.4c). They are given explicitly in Brabston (1974).

The drag coefficient C_D is defined by

$$C_D = D/\pi\rho U^2 a^2,$$

where D is the drag on the sphere and ρ is the fluid density. The drag is given in terms of an integral over any surface surrounding the sphere. Using for the surface the sphere ξ = constant, C_D can be calculated as a rather complicated sum involving all the $f_n(\xi)$, $g_n(\xi)$, $f_n'(\xi)$ and $g_n'(\xi)$. This expression simplifies considerably when $\xi = 0$ and the drag is computed on the surface of the bubble.

The pressure coefficient is

$$k(\theta) = (p^*(0,\theta) - p_\infty^*)/\tfrac{1}{2}\rho U^2,$$

where $p^*(0,\theta)$ is the dimensional pressure on the sphere and p_∞^* is the dimensional free-stream pressure. Explicit formulae for $C_D(\xi)$ and $k(\theta)$ are given in Brabston (1974).

3. Numerical solution

We truncate the infinite system (2.6) with (2.7) at N terms; that is we set

$$f_n(\xi) \equiv g_n(\xi) \equiv 0 \quad \text{for all} \quad n > N. \tag{3.1}$$

The truncated system is written as a finite first-order system:

$$\mathbf{F}'(\xi) = \mathbf{A}(\xi)\, \mathbf{F}(\xi) + \mathbf{N}(\xi, \mathbf{F}), \tag{3.2}$$

where

$$\mathbf{F}(\xi) \equiv \begin{pmatrix} \mathbf{f}'(\xi) \\ \mathbf{g}'(\xi) \\ \mathbf{f}(\xi) \\ \mathbf{g}(\xi) \end{pmatrix}, \quad \mathbf{N}(\xi, \mathbf{F}) \equiv \begin{pmatrix} 0 \\ \mathbf{r} \\ 0 \\ 0 \end{pmatrix}$$

are $4N$-component vectors, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{r}$, etc., being N-component vectors with components $f_n(\xi)$, $g_n(\xi)$, $r_n(\xi)$, etc.,

$$\mathbf{A}(\xi) \equiv \begin{pmatrix} 0 & 0 & \mathbf{K}_1 & \mathbf{K}_2(\xi) \\ 0 & -\mathbf{I} & 0 & \mathbf{K}_3 \\ \mathbf{I} & 0 & 0 & 0 \\ 0 & \mathbf{I} & 0 & 0 \end{pmatrix}$$

is $4N \times 4N$, $\mathbf{I}$ being $N \times N$, while $\mathbf{K}_1$, $\mathbf{K}_2$ and $\mathbf{K}_3$ are $N \times N$ diagonal matrices with diagonal elements

$$(n+\tfrac{1}{2})^2, \quad n(n+1)e^{\frac{1}{2}\xi}, \quad n(n+1)$$

respectively. The boundary conditions are written as

$$\mathbf{B}_0 \mathbf{F}(0) = 0, \quad \mathbf{B}_1 \mathbf{F}(\xi_\infty) = \mathbf{\Gamma}_\infty, \tag{3.3}$$

where

$$\mathbf{\Gamma}_\infty = \begin{pmatrix} f_1^\infty(\xi_\infty) \\ \vdots \\ g_N^\infty(\xi_\infty) \end{pmatrix}, \quad \mathbf{B}_0 = \begin{pmatrix} 0 & 0 & \mathbf{I} & 0 \\ \mathbf{I} & 0 & 0 & -\tfrac{1}{2}\mathbf{K}_3 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & \mathbf{I} \end{pmatrix}.$$

Equations (3.2) and (3.3) form a $4N$-component first-order nonlinear system with two-point separated end conditions.

To solve the system (3.2) with (3.3), we use the centred Euler or 'box' scheme studied by Keller (1974). Thus we introduce a net on $[0, \xi_\infty]$ defined by

$$\xi_0 = 0, \quad \xi_j = jh, \quad j = 1, 2, \ldots, J, \quad h = \xi_\infty/J.$$

Then the centred Euler scheme for (3.2) and (3.3) is

$$h^{-1}[\mathbf{F}_j - \mathbf{F}_{j-1}] = \mathbf{A}(\xi_{j-\frac{1}{2}})\tfrac{1}{2}[\mathbf{F}_j + \mathbf{F}_{j-1}] + \mathbf{N}(\xi_{j-\frac{1}{2}}, \tfrac{1}{2}[\mathbf{F}_j + \mathbf{F}_{j-1}]), \tag{3.4a}$$

$$\mathbf{B}_0 \mathbf{F}_0 = 0, \quad \mathbf{B}_1 \mathbf{F}_J = \mathbf{\Gamma}_\infty, \tag{3.4b}$$

where $\mathbf{F}_j$ is the numerical approximation to $\mathbf{F}(\xi_j)$ and $\xi_{j-\frac{1}{2}} \equiv \xi_j - \frac{1}{2}h$. The system of $4N(J+1)$ nonlinear difference equations (3.4) can be written as

$$\mathbf{\Phi}(\mathbf{F}) = 0, \tag{3.5}$$

where

$$\mathbf{F} \equiv \begin{pmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_J \end{pmatrix}, \quad \mathbf{\Phi}(\mathbf{F}) \equiv \begin{pmatrix} \mathbf{B}_0 \mathbf{F}_0 \\ \mathbf{F}_1 - \mathbf{F}_0 - \frac{1}{2}h\mathbf{A}(\xi_{\frac{1}{2}})[\mathbf{F}_1 + \mathbf{F}_0] - h\mathbf{N}(\xi_{\frac{1}{2}}, \frac{1}{2}[\mathbf{F}_1 + \mathbf{F}_0]), \\ \vdots \\ \mathbf{F}_J - \mathbf{F}_{J-1} - \frac{1}{2}h\mathbf{A}(\xi_{J-\frac{1}{2}})[\mathbf{F}_J + \mathbf{F}_{J-1}] \\ \qquad - h\mathbf{N}(\xi_{J-\frac{1}{2}}, \frac{1}{2}[\mathbf{F}_J + \mathbf{F}_{J-1}]) \\ \mathbf{B}_1 \mathbf{F}_J - \mathbf{\Gamma}_\infty \end{pmatrix}.$$

To solve this system we use Newton's method with the sequence of iterates $\{\mathbf{F}^{(\nu)}\}$ written as

$$\mathbf{F}^{(\nu+1)} = \mathbf{F}^{(\nu)} + \delta\mathbf{F}^{(\nu)}. \tag{3.6}$$

The corrections $\delta\mathbf{F}^{(\nu)}$ are defined by using (3.6) in (3.5) and linearizing to get

$$\mathbf{T}^{(\nu)}\delta\mathbf{F}^{(\nu)} \equiv (\partial\mathbf{\Phi}^{(\nu)}/\partial\mathbf{F})\,\delta\mathbf{F}^{(\nu)} = -\mathbf{\Phi}(\mathbf{F}^{(\nu)}). \tag{3.7}$$

Here $\mathbf{T}^{(\nu)} \equiv \partial\mathbf{\Phi}^{(\nu)}/\partial\mathbf{F}$ is the $4N(J+1)$-order Jacobian matrix evaluated at $\mathbf{F} = \mathbf{F}^{(\nu)}$. It has block tridiagonal form and thus $\delta\mathbf{F}^{(\nu)}$ can be efficiently computed by a variety of direct factorization methods. Under very general conditions it can be proved that the Newton iterates converge quadratically; that is,

$$\|\delta\mathbf{F}^{(\nu+1)}\| \leqslant K\|\delta\mathbf{F}^{(\nu)}\|^2.$$

Thus the convergence of this method is fast and can be observed in the calculations.

Owing to the two-point nature of the difference scheme, the separated form of the boundary conditions and the order in which we have written the equations in (3.5) the matrix $\mathbf{T}$ has the block tridiagonal form

$$\mathbf{T} = \begin{pmatrix} \mathbf{A}_1 & \mathbf{C}_1 & & & \\ \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_2 & 0 & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ 0 & & \cdot & \cdot & \mathbf{C}_J \\ & & & \mathbf{B}_{J+1} & \mathbf{A}_{J+1} \end{pmatrix}, \tag{3.8}$$

where each of the block matrices $\mathbf{A}_j$, $\mathbf{B}_j$ and $\mathbf{C}_j$ is $4N \times 4N$. Since the boundary-condition matrices $\mathbf{B}_0$ and $\mathbf{B}_1$ are $2N \times 4N$, the $\mathbf{C}_j$'s all have zeros in their upper $2N$ rows as do the $\mathbf{B}_j$'s in their lower $2N$ rows.

The basic block or band factorization methods for solving (3.7) all become inaccurate as R and N are increased. Residual corrections fail to improve the results significantly. Hence we turn to a more sophisticated factorization motivated by the 'parallel shooting' method for differential equations studied by Keller (1968).

Our factorization consists of writing $\mathbf{T}$ as the product of two matrices of the form

$$\mathbf{T} = \mathbf{T}_1\mathbf{T}_2 \equiv \begin{pmatrix} \mathbf{L}_1 & 0 \\ 0 & \mathbf{U}_2 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 & \boldsymbol{\delta} & 0 \\ 0 & \boldsymbol{\gamma} & \mathbf{L}_2 \end{pmatrix}. \tag{3.9}$$

Here $\mathbf{L}_1$ is a $4Nk$-order square-block lower diagonal matrix, $\mathbf{U}_1$ is a $4Nk$-block upper diagonal matrix, $\mathbf{U}_2$ is a $4(J+1-k)$-block upper triangular matrix, $\mathbf{L}_2$ is a $4N(J+1-k)$-block lower triangular matrix and $\boldsymbol{\gamma}$ and $\boldsymbol{\delta}$ are $4N$-order matrices. The elements of $\mathbf{T}_1$ and $\mathbf{T}_2$ can be determined explicitly once the forms of the diagonal matrices in, say, $\mathbf{T}_1$ are specified. For our computations, we chose the diagonal blocks of $\mathbf{T}_1$ to be the $4N$-order identity matrix. Brabston (1974) gives the complete recursion relations for solving (3.7) using the factorization (3.9). These relations are very similar to those for the simpler factorizations given by Keller (1974).

Given an initial estimate $\mathbf{F}^{(0)}$, which we take as the solution for the next lowest R value, we use (3.6), (3.7) and the factorization (3.9) to generate successive

R	N	J	$\delta C_D^{(1)}$	$\lVert\delta \mathbf{F}^{(1)}\rVert_\infty$	$\delta C_D^{(2)}$	$\lVert\delta \mathbf{F}^{(2)}\rVert_\infty$	$\delta C_D^{(3)}$	$\lVert\delta \mathbf{F}^{(3)}\rVert_\infty$
0·1	6	31	77·61	2335·0	0·4575	3·789	$0{\cdot}6256\times10^{-3}$	$0{\cdot}3388\times10^{-2}$
0·5	6	31	0·6254	4·334	$0{\cdot}1053\times10^{-2}$	$0{\cdot}2635\times10^{-1}$	0·0	$0{\cdot}3812\times10^{-3}$
5	6	31	0·3545	4·375	$0{\cdot}4194\times10^{-3}$	0·1238	$0{\cdot}6986\times10^{-5}$	$0{\cdot}3738\times10^{-3}$
20	16	31	0·1567	2·340	$0{\cdot}1559\times10^{-1}$	1·161	$0{\cdot}5717\times10^{-3}$	$0{\cdot}4582\times10^{-1}$
60	20	61	$0{\cdot}7448\times10^{-1}$	1·799	$0{\cdot}7151\times10^{-2}$	1·193	$0{\cdot}9614\times10^{-4}$	0·3662
120	20	61	0·1130	5·492	$0{\cdot}1915\times10^{-2}$	1·552	$0{\cdot}2867\times10^{-4}$	$0{\cdot}8015\times10^{-1}$
200	30	61	$0{\cdot}8336\times10^{-1}$	62·07	$0{\cdot}2049\times10^{-2}$	80·53		

TABLE 1. Convergence of $\delta C_D^{(\nu)}$ and $\lVert\delta \mathbf{F}^{(\nu)}\rVert$

Newton iterates. For 'small' R the rigid-sphere solution can be used for the initial guess.

Keller (1974) shows that, under reasonable conditions, the error incurred by using the difference scheme has the expansion

$$\mathbf{F}(\xi_j)-\mathbf{F}_j = h^2\mathbf{e}(\xi_j)+O(h^4). \tag{3.10}$$

Here $\mathbf{e}(\xi)$ is independent of h, so that Richardson extrapolation is valid. In this extrapolation, we compute $\mathbf{F}_{j,1}$ and $\mathbf{F}_{j,2}$ (approximations to $\mathbf{F}(\xi_j)$) using mesh spacings h and $\frac{1}{2}h$, respectively. Then, defining

$$\hat{\mathbf{F}}_j = \tfrac{4}{3}\mathbf{F}_{j,2}-\tfrac{1}{3}\mathbf{F}_{j,1}, \tag{3.11}$$

we have

$$\mathbf{F}(\xi_j)-\hat{\mathbf{F}}_j = O(h^4).$$

Thus $\hat{\mathbf{F}}_j$ is a higher-order accurate approximation to the solution.

4. Error evaluation

To assess the accuracy of our computed solution, we note that there are five sources of error: (*a*) iteration error in solving (3.4); (*b*) discretization error, (*c*) series truncation error, (*d*) outer boundary condition error, (*e*) round-off error.

We checked to see that Newton's method converges quadratically by observing the drag coefficient $C_D^{(\nu)}$, computed on the sphere surface, for each iteration. Iterations were continued until we could establish that

$$|C_D^{(\nu+1)}-C_D^{(\nu)}| \leqslant 3|C_D^{(\nu)}-C_D^{(\nu-1)}|^2. \tag{4.1}$$

The convergence of $\delta C_D^{(\nu)}$ and of $\delta\mathbf{F}^{(\nu)}$ are shown for a range of Reynolds numbers in table 1. For each R except $R = 200$ we also computed until

$$|\delta C_D^{(\nu)}| = |C_D^{(\nu)}-C_D^{(\nu-1)}| < 10^{-3}.$$

The finite-difference discretization error can be reduced by using Richardson extrapolation as described in §3. One important check on the numerical solution is to see how the computed drag coefficient varies with the radius. This variation is shown in table 2. We observe that Richardson extrapolation improves the drag constancy quite significantly. Note however the large deviation of $C_D(\xi)$ from $C_D(0)$ as $\xi\to\xi_\infty$ for $R = 200$. We believe this to be due to ill conditioning in

	$R = 0{\cdot}5$, $N = 6$		$R = 40$, $N = 20$		$R = 60$, $N = 20$		$R = 200$, $N = 30$	
ξ	I	II	I	II	I	II	I	II
0	16·24	16·85	0·3963	0·4156	0·2967	0·3001	0·1012	0·0985
0·32667	16·14	16·85	0·3993	0·4153	0·2972	0·3000	0·1015	0·0995
0·65333	16·05	16·85	0·3850	0·4156	0·2938	0·3001	0·1007	0·0995
0·98000	15·95	16·84	0·3729	0·4158	0·2914	0·3001	0·1004	0·0996
1·3067	15·85	16·84	0·3630	0·4158	0·2895	0·3001	0·1001	0·0997
1·6333	15·76	16·84	0·3546	0·4157	0·2879	0·3001	0·0999	0·0998
1·9600	15·67	16·84	0·3470	0·4155	0·2864	0·3001	0·0998	0·1000
2·2867	15·56	16·84	0·3401	0·4153	0·2850	0·3001	0·1016	0·1038
2·6133	15·45	16·84	0·3335	0·4150	0·2837	0·3001	0·1012	0·1035
2·9400	15·33	16·84	0·3273	0·4146	0·2825	0·3000	0·1012	0·1030
3·2667	15·18	16·83	0·3213	0·4143	0·2812	0·3000	0·1017	0·1027
3·5933	15·01	16·83	0·3162	0·4136	0·2800	0·3000	0·1033	0·1063
3·9200	14·82	16·83	0·3134	0·4124	0·2788	0·2999	0·1072	0·1141
4·2467	14·61	16·82	0·3087	0·4115	0·2776	0·2987	0·1098	0·1313
4·5733	14·39	16·81	0·3036	0·4109	0·2763	0·2992	−0·0216	−0·0971
4·9000	14·23	16·80	0·2982	0·4103	0·2750	0·2996	−0·1083	−0·2717

TABLE 2. $C_D(\xi)$ calculated with and without Richardson extrapolation. I, without extrapolation; II, with extrapolation.

R	N	J	Richardson
0·1	6	60	Yes
0·5	6	60	Yes
1	6	60	Yes
5	6	60	Yes
10	16	30	No
20	20	30	No
40	20	60	Yes
60	20	90	Yes
120	30	90	Yes
200	30	90	Yes

TABLE 3. N and J used for various R

solving (3.7) for the Newton iterates. Indeed the same phenomenon occurred for smaller R and that was the motivation for introducing the 'parallel-shooting' factorization (3.9). We have not tried to eliminate this growth of $C_D(\xi)$ for $R = 200$.

The error due to truncating the series can only be estimated empirically since we have no analytic bound on the remainder. Clearly one expects that the number N of terms should be increased with R. We did not do any serious testing of the effect of varying N for larger Reynolds numbers as the computation time varies as N^3. Our modest tests have general agreement with the truncations used by Dennis & Walker (1971) for rigid spheres. Table 3 shows the largest value of N and J used for each Reynolds number.

The error due to imposing the outer boundary conditions at a finite distance from the sphere can be estimated by computing solutions with different values of ξ_∞. We computed solutions with ξ_∞ equal to π and 4·9 (the value used by Dennis & Walker), and the results were substantially different. The greater distance was used in all of our calculations. This may not be a sufficiently large value for $R > 200$. More testing and a better understanding of this point would be of great value. This error may be related to the growth of $C_D(\xi)$ as $\xi \to \xi_\infty$ but we have no strong evidence for this in our present computations.

It is far from clear that the Oseen expansion for a rigid sphere as imposed in (2.4*c*) is valid for all Reynolds numbers. Indeed the correct condition for flow about a gas bubble must lie between that for a rigid sphere and the free stream (i.e. no obstruction). Thus we have computed the solutions using free-stream conditions imposed at $\xi_\infty = 4{\cdot}9$ for $R = 5$ and $R = 120$. The resulting C_D values computed on the sphere surface were altered by about 0·01 %.

The round-off errors are a function of the computer used to perform the calculations. Our computations were done on an IBM 370/155 in single precision, which gives about seven significant digits. Round-off error compounded by the ill conditioning of the linear system (3.7) caused a lack of quadratic convergence in Newton's method for the higher Reynolds numbers. Upon using the 'parallel shooting' factorization (3.9) of $\mathbf{T}$, Newton's method converged quadratically. This convergence and the demonstrated accuracy given by Richardson extrapolation indicate that round-off error is not a substantial factor in most of our results. However we believe that it does contribute to the variation in $C_D(\xi)$ at $R = 200$.

5. Results and assessment

Table 3 shows the Reynolds numbers for which our calculations were performed along with the numbers of series terms and maximum numbers of mesh points used. Richardson extrapolation was used for all Reynolds numbers except $R = 10$ and $R = 20$.

One of the main purposes of our work was to check the theoretical result of Moore (1963) for the drag on the bubble at high Reynolds numbers. Batchelor (1967, p. 368) gives the drag coefficient as

$$C_D = 24/R. \tag{5.1}$$

Moore gives the higher-order approximation for this as

$$\begin{aligned} C_D &= \frac{24}{R}\left(1 - \frac{4 \times 2^{\frac{1}{3}}(6 \times 3^{\frac{1}{2}} + 5 \times 2^{\frac{1}{3}} - 14)}{5\pi^{\frac{1}{3}} R^{\frac{1}{2}}} + O(R^{-\frac{5}{6}})\right) \\ &= \frac{24}{R}\left(1 - \frac{2{\cdot}2107}{R^{\frac{1}{2}}} + O(R^{-\frac{5}{6}})\right). \end{aligned} \tag{5.2}$$

Our calculations bear out Moore's result very well. Table 4 shows our computed value of C_D, Moore's value, the first-term expansion value and the difference between our value and Moore's value as a percentage of the latter.

At low Reynolds numbers the asymptotic formula for the drag coefficient is

$$C_D = 8/R.$$

R	Present $C_D(0)$	Moore's C_D	Difference (%)	$24/R$
10	1·175	0·7222	62·7	2·4
20	0·6810	0·6068	12·2	1·2
40	0·4156	0·3903	6·47	0·6
60	0·3001	0·2858	5·00	0·4
120	0·1647	0·1596	3·18	0·2
200	0·0985	0·1012	2·71	0·12

TABLE 4. Drag coefficient for present calculations and Moore's theory

R	$8/R$	Present $C_D(0)$	Difference (%)
0·1	80	80·83	1·04
0·5	16	16·85	5·31
1·0	8	8·795	9·95
5·0	1·6	2·184	36·3

TABLE 5. Drag coefficients for low Reynolds numbers

Table 5 shows this theoretical value of the drag coefficient, our calculated value and the difference (as a percentage of the theoretical value) for the lower Reynolds numbers. From tables 4 and 5 we thus see that our computed values for $0{\cdot}5 \leqslant R \leqslant 40$ cover a range in which no accurate approximate formula is known.

Other physical parameters of interest are the vorticity and pressure coefficient (defined in §2) on the bubble surface. These are plotted for the higher Reynolds numbers in figures 1 (*a*) and (*b*). The transverse velocity on the bubble is given by $v(0,\theta) = \frac{1}{2}\zeta(0,\theta)$. One physical phenomenon of special interest is that of flow separation behind the bubble. Levich (1962, p. 445) states that the separation zone extends only 2° on either side of the line of symmetry at $R = 1250$. Thus it was no surprise that we did not find evidence of separation. However, it should be noted that the pressure profiles at $R = 120$ and 200 are rather far from the symmetric profiles of the inviscid theory.

The major portion of the computation time is required to solve the linear system (3.7). The number of operations involved, and hence the time, is proportional to N^3J. For $N = 20$ and $J = 60$, the computing time was approximately 2100 s for one Newton iterate on an IBM 370/155. Since both N and J must increase with Reynolds number we are very close to the limit of what can reasonably be done with this method on our present machine.

We have computed the drag on rigid spheres for $R = 0{\cdot}1$, 0·5, 1, 5, 10. Our results agree with those of Dennis & Walker (1971) to three or four digits.

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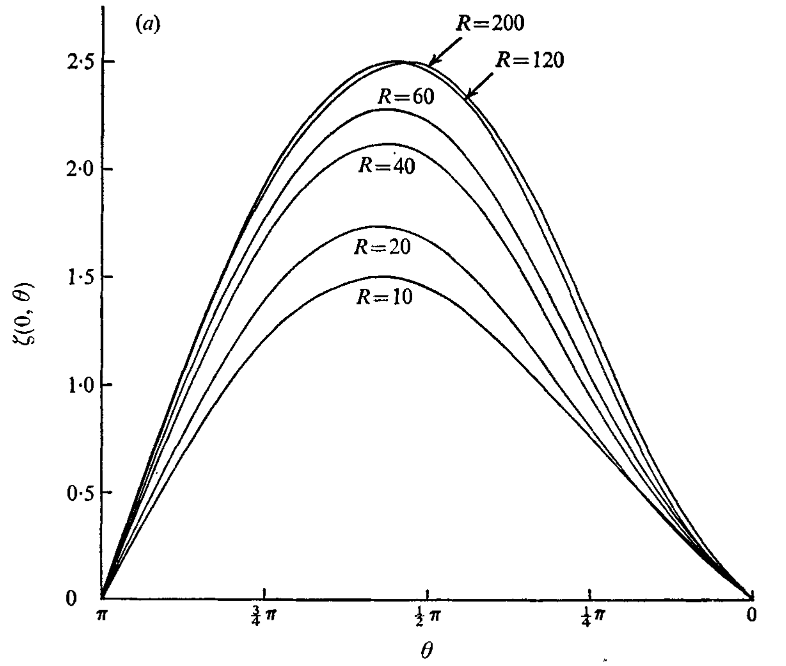

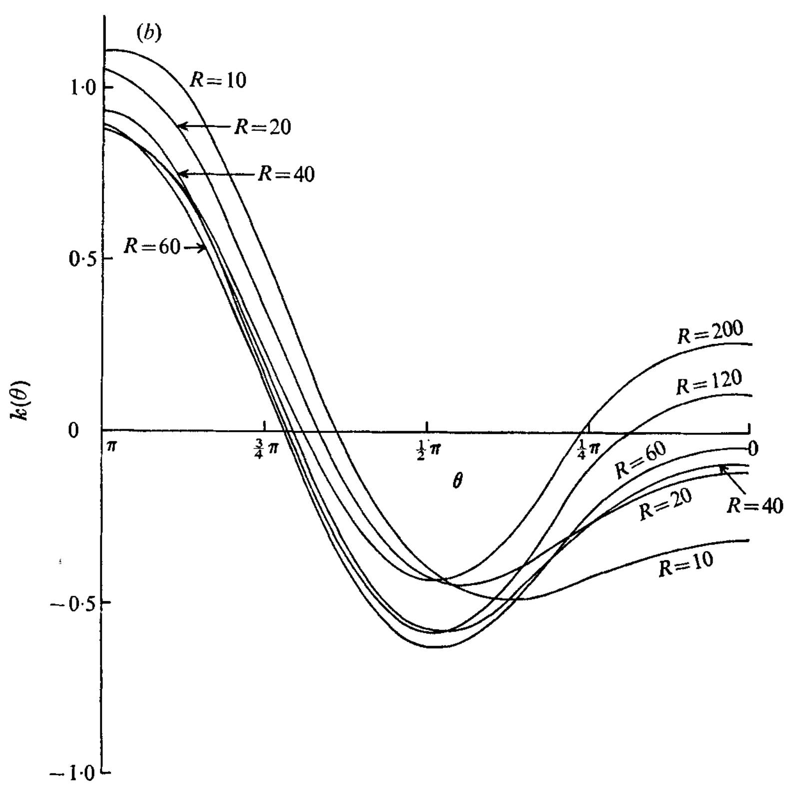

FIGURE 1. (a) Vorticity and (b) pressure coefficients on the bubble surface.

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